Dirac(-Pauli), Fokker-Planck equations and exceptional Laguerre polynomials

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Abstract

An interesting discovery in the last two years in the field of mathematical physics has been the exceptional X_ℓ Laguerre and Jacobi polynomials. Unlike the well-known classical orthogonal polynomials which start with constant terms, these new polynomials have lowest degree $\ell = 1, 2, \ldots,$ and yet they form complete set with respect to some positive-definite measure. While the mathematical properties of these new X_ℓ polynomials deserve further analysis, it is also of interest to see if they play any role in physical systems. In this paper we indicate some physical models in which these new polynomials appear as the main part of the eigenfunctions. The systems we consider include the Dirac equations coupled minimally and non-minimally with some external fields, and the Fokker-Planck equations. The systems presented here have enlarged the number of exactly solvable physical systems known so far.

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I. INTRODUCTION

The past two years have witnessed some interesting developments in the area of exactly solvable models in quantum mechanics: the number of exactly solvable shape-invariant models has been greatly increased owing to the discovery of new types of orthogonal polynomials, called the exceptional X_ℓ polynomials. Two families of such polynomials, namely, the Laguerre- and Jacobi-type X_1 polynomials, corresponding to $\ell = 1$, were first proposed by Gómez-Ullate et al. in [1], within the Sturm-Liouville theory, as solutions of second-order eigenvalue equations with rational coefficients. Unlike the classical orthogonal polynomials, these new polynomials have the remarkable properties that they still form complete set with respect to some positive-definite measure, although they start with a linear polynomials instead of a constant. The results in [1] were then reformulated by Quesne in the framework of quantum mechanics and shape-invariant potentials, first in [2] by the point canonical transformation method, and then in [3] by supersymmetric (SUSY) method [4] (or the Darboux-Crum transformation [5]). Soon after these works, such kind of exceptional polynomials were generalized by Odake and Sasaki to all integral $\ell = 1, 2, \ldots$ [6] (the case of $\ell = 2$ was also discussed in [3]). By construction these new polynomials satisfy the Schrödinger equation and yet they start at degree $\ell > 0$ instead of the degree zero constant term. Thus they are not constrained by Bochner's theorem [7], which states that the orthogonal polynomials (starting with degree 0) satisfying a second order differential equations can only be the classical orthogonal polynomials, i.e., the Hermite, Laguerre, Jacobi and Bessel polynomials.

Later, equivalent but much simpler looking forms of the Laguerre- and Jacobi-type X_ℓ polynomials than those originally presented in [6] were given in [8]. These nice forms were derived based on an analysis of the second order differential equations for the X_ℓ polynomials within the framework of the Fuchsian differential equations in the entire complex x-plane. They allow us to study in-depth some important properties of the X_ℓ polynomials, such as the actions of the forward and backward shift operators on the X_ℓ polynomials, Gram-Schmidt orthonormalization for the algebraic construction of the X_ℓ polynomials, Rodrigues formulas, and the generating functions of these new polynomials.

Recently, in [9] the X_1 Laguerre polynomials in [1] are generalized to the X_ℓ Laguerre polynomials with higher ℓ based on the Darboux-Crum transformation. Then in [10] such

transformation was successfully employed to generate the X_ℓ Jabobi as well as the X_ℓ Laguerre polynomials as given in [8].

While the mathematical properties of these new X_ℓ polynomials deserve further analysis, it is also of interest to see if they play any role in physical systems. It is the purpose of the present work to indicate some physical models that involve these new polynomials. We shall be mainly concerned with the X_ℓ Laguerre polynomials for clarity of presentation. Models that are linked with the X_ℓ Jacobi polynomials will be briefly mentioned at the end.

The plan of the paper is as follows. First we review the deformed radial oscillators associated with the exceptional X_ℓ Laguerre polynomials in Sect. II. In Sect. III the Dirac equation minimally coupled with an external magnetic field is considered. We present the forms of the vector potentials such that the eigenfunctions of the Dirac equation are related to the X_ℓ Laguerre polynomials. Dirac equations with non-minimal couplings are then mentioned in Sect. IV, and the Fokker-Planck equations are considered in Sect. V. Sect. VI concludes the paper.

II.   EXCEPTIONAL X_ℓ LAGUERRE POLYMOMIALS

Consider a generic one-dimensional quantum mechanical system described by a Hamiltonian $H = -d^2/dx^2 + V_0(x)$. Suppose the ground state is given by the wave function $\phi_0(x)$ with zero energy: $H\phi_0 = 0$. By the well known oscillation theorem ϕ_0 is nodeless, and thus can be written as $\phi_0 \equiv e^{W_0(x)}$, where $W_0(x)$ is a regular function of x. This implies that the function $W_0(x)$ completely determines the potential V_0 : $V_0 = W_0'^2 + W_0''$ (the prime here denotes derivatives with respect to x). Thus $W_0(x)$ is sometimes called a prepotential. Consequently, the Hamiltonian can be factorized as $H \equiv H_0^{(+)} = A_0^- A_0^+$, with $A_0^\pm \equiv \pm d/dx - W_0'$. It is trivial to verify $A_0^+ \phi_0(x) = 0$.

It is a remarkable fact that most well-known one-dimensional exactly solvable systems possess a property called shape invariance [11]. This means the Hamiltonian $H_0^{(-)}$, defined by $H_0^{(-)} = A_0^+ A_0^-$ with potential $W_0'^2 - W_0''$, is related to $H_0^{(+)}$ by the relation $H_0^{(-)}(\boldsymbol{\lambda}) = H_0^{(+)}(\boldsymbol{\lambda} + \boldsymbol{\delta}) + \mathcal{E}_1(\boldsymbol{\lambda})$. Here $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots)$ is a set of parameters of the Hamiltonian $H = H(\boldsymbol{\lambda})$, $\boldsymbol{\delta}$ is a certain shift of these parameters, and $\mathcal{E}_1(\boldsymbol{\lambda})$ is some function of $\boldsymbol{\lambda}$. The mapping that relates the potentials $H_0^{(+)}$ and $H_0^{(-)}$ is called the Darboux-Crum transformation [5] ($H_0^{(+)}$ and $H_0^{(-)}$ are also called SUSY partners in the context of SUSY

quantum mechanics [4]). Shape invariance is a sufficient condition [12] that enables one to determine the eigenvalues and the corresponding eigenfunctions of $H_0^{(+)}(\boldsymbol{\lambda})$ exactly (see [4] for details). Specifically, we have

$$\mathcal{E}_0(\boldsymbol{\lambda}) = 0, \quad \mathcal{E}_m(\boldsymbol{\lambda}) = \sum_{k=0}^{n-1} \mathcal{E}_1(\boldsymbol{\lambda} + k\boldsymbol{\delta}), \tag{1}$$

$$\phi_m(x; \boldsymbol{\lambda}) \propto A_0^-(\boldsymbol{\lambda}) A_0^-(\boldsymbol{\lambda} + \boldsymbol{\delta}) \cdots A_0^-(\boldsymbol{\lambda} + (m-1)\boldsymbol{\delta}) \times e^{W_0(x; \boldsymbol{\lambda} + m\boldsymbol{\delta})}, \quad m = 1, 2, \ldots \tag{2}$$

The new shape-invariant systems related to the exceptional polynomials are determined by certain prepotentials W_ℓ ($\ell = 1, 2, \ldots$), which are obtained by deforming some shape invariant prepotentials W_0 [6]. The $\ell = 0$ case corresponds to the original system. Three families of such exactly solvable deformed systems were presented in [6], which correspond to deforming the radial oscillator and the trigonometric/hyperbolic Darboux-Pöschl-Teller potentials in terms of their respective eigenfunctions.

For the purpose of this paper, we shall only consider the exceptional X_ℓ Laguerre polynomials, which appear in the deformed radial oscillator potentials. The original radial oscillator potential is generated by the prepotential

$$W_0(x; g) = -\frac{\omega x^2}{2} + g \log x, \quad 0 < x < \infty. \tag{3}$$

Here $\boldsymbol{\lambda} = g > 0$ and $\omega > 0$. The Hamiltonian is

$$H_0^{(+)}(g) = A^-(g) A^+(g) = -\frac{d^2}{dx^2} + \omega^2 x^2 + \frac{g(g-1)}{x^2} - (2g+1)\omega. \tag{4}$$

This potential is shape-invariant with shift parameter $\boldsymbol{\delta} = 1$ and $\mathcal{E}_1(g) = 4\omega$: $H_0^{(-)}(g) = H_0^{(+)}(g+1) + 4\omega$. The eigen-energies and eigenfunctions are ($n = 0, 1, 2, \ldots$)

$$\mathcal{E}_n(g) = 4n\omega; \quad \phi_n(x; g) = e^{W_0(x; g)} P_n(\eta; g), \quad P_n(\eta; g) \equiv L_n^{(g - \frac{1}{2})}(\eta), \tag{5}$$

where $\eta(x) \equiv \omega x^2$ is one of the so-called sinusoidal coordinates [13].

Below we shall present radial oscillators related to the X_ℓ Laguerre polynomials. We first treat them as systems deformed from the original radial oscillator by appropriate deforming functions. Then we show how they can also be considered as the Darboux-Crum (or SUSY) partner of the original radial oscillator.

A. Deformed radial oscillators

The Hamiltonian $H_\ell^{(+)}(g)$ of the deformed radial oscillator is

$$H_\ell^{(+)}(g) = -\frac{d^2}{dx^2} + W_\ell'^2(x;g) + W_\ell''(x;g), \tag{6}$$

where W_ℓ is given by

$$W_\ell(x;g) = -\frac{\omega x^2}{2} + (g+\ell)\log x + \log\frac{\xi_\ell(\eta; g+1)}{\xi_\ell(\eta; g)}. \tag{7}$$

Here $\xi_\ell(\eta; g)$ is a deforming function. It turns out there are two possible sets of deforming functions $\xi_\ell(\eta; g)$, thus giving rise to two sets of infinitely many exceptional Laguerre polynomials, termed L1 and L2 type [6, 8]. These ξ_ℓ are given by

$$\xi_\ell(\eta; g) = \begin{cases} L_\ell^{(g+\ell-\frac{3}{2})}(-\eta) & : \text{L1} \\ L_\ell^{(-g-\ell-\frac{1}{2})}(\eta) & : \text{L2}. \end{cases} \tag{8}$$

For both types of ξ_ℓ, the eigen-energies are $\mathcal{E}_{\ell,n}(g) = \mathcal{E}_n(g+\ell) = 4n\omega$, which are independent of g and ℓ. Hence the deformed radial oscillator is iso-spectral to the ordinary radial oscillator. The eigenfunctions are given by

$$\phi_{\ell,n}(x;g) = \frac{e^{-\frac{1}{2}\omega x^2} x^{g+\ell}}{\xi_\ell(\eta; g)} P_{\ell,n}(\eta; g), \tag{9}$$

where the corresponding exceptional Laguerre polynomials $P_{\ell,n}(\eta; g)$ ($\ell = 1, 2, \ldots$, $n = 0, 1, 2, \ldots$) can be expressed as a bilinear form of the original Laguerre polynomials and the deforming polynomials, as given in [8]:

$$P_{\ell,n}(\eta; g) = \begin{cases} \xi_\ell(\eta; g+1)P_n(\eta; g+\ell-1) - \xi_\ell(\eta; g)\partial_\eta P_n(\eta; g+\ell-1) & : \text{L1} \\ (n+g+\frac{1}{2})^{-1}\big((g+\frac{1}{2})\xi_\ell(\eta; g+1)P_n(\eta; g+\ell+1) \\ \qquad\qquad + \eta\xi_\ell(\eta; g)\partial_\eta P_n(\eta; g+\ell+1)\big) & : \text{L2}. \end{cases} \tag{10}$$

The X_ℓ polynomials $P_{\ell,n}(\eta; g)$ are degree $\ell + n$ polynomials in η and start at degree ℓ: $P_{\ell,0}(\eta; g) = \xi_\ell(\eta; g+1)$. They are orthogonal with respect to certain weight functions, which are deformations of the weight function for the Laguerre polynomials (for details, see [8]).

B. Darboux-Crum pairs

As mentioned in Sect. 1, the new exceptional orthogonal polynomials have recently been re-derived from the corresponding ordinary polynomials based on the Darboux-Crum trans-

formation [9, 10]. Below we shall reconsider the case of the deformed radial oscillator, following the approach of [10].

Instead of finding the prepotential W_ℓ that gives $H_0^{(+)}$ as the deformed oscillator, as was done in the previous subsection, one determines a new prepotential W_ℓ so that it is $H_0^{(-)}$ that gives the deformed oscillator while $H_0^{(+)}$ is related to the ordinary oscillator.

The new prepotential $W_\ell(x; g)$ are [10]

$$
W_\ell(x; g) = \begin{cases} \frac{1}{2}\omega x^2 + (g + \ell - 1)\log x + \log \xi_\ell(\eta(x); g), & g > 1/2, \quad : \text{L1} \\ -\frac{1}{2}\omega x^2 - (g + \ell)\log x + \log \xi_\ell(\eta(x); g), & g > -1/2, \quad : \text{L2} \end{cases}
\tag{11}
$$

where $\xi_\ell(\eta; g)$ are as given in (8). Now consider the pairs of Hamiltonians $\mathcal{H}_\ell^{(+)}(g)$ and $\mathcal{H}_\ell^{(-)}(g)$ ($\ell = 1, 2, \ldots$) defined by

$$
\mathcal{H}_\ell^{(+)}(g) = \mathcal{A}_\ell^-(g)\mathcal{A}_\ell^+(g), \quad \mathcal{H}_\ell^{(-)}(g) = \mathcal{A}_\ell^+(g)\mathcal{A}_\ell^-(g),
\tag{12}
$$

$$
\mathcal{A}_\ell^\pm(g) = \pm\frac{d}{dx} - W_\ell'(x; g).
\tag{13}
$$

Using the differential equation for the Laguerre polynomial, the Hamiltonians $\mathcal{H}_\ell^{(+)}(g)$ can be shown to be related to that of the radial oscillator $H_0^{(+)}(g)$ in (4) as

$$
\mathcal{H}_\ell^{(+)}(g) = \begin{cases} H_0^{(+)}(g + \ell - 1) + 2(2g + 4\ell - 1)\omega & : \text{L1} \\ H_0^{(+)}(g + \ell + 1) + 2(2g + 1)\omega & : \text{L2}. \end{cases}
\tag{14}
$$

The partner Hamiltonians are found to equal to the Sasaki-Odake Hamiltonian $H_\ell^{(+)}(g)$ in (6) up to additive constants:

$$
\mathcal{H}_\ell^{(-)}(g) = \begin{cases} H_\ell^{(+)}(g) + 2(2g + 4\ell - 1)\omega & : \text{L1} \\ H_\ell^{(+)}(g) + 2(2g + 1)\omega & : \text{L2}. \end{cases}
\tag{15}
$$

It is shown in [10] that the partner Hamiltonian $\mathcal{H}_\ell^{(-)}(g)$ are exactly iso-spectral to the radial oscillator Hamiltonian $\mathcal{H}_\ell^{(+)}(g)$, which have the following eigenvalues and the corresponding eigenfunctions ($\eta = \omega x^2$):

L1: $\quad \mathcal{E}_{\ell,n}^{(+)}(g) = 4\left(n + g + 2\ell - \frac{1}{2}\right)\omega, \quad \phi_{\ell,n}^{(+)}(x; g) = e^{-\frac{1}{2}\omega x^2}x^{g+\ell-1}L_n^{(g+\ell-\frac{3}{2})}(\eta),$ (16)

L2: $\quad \mathcal{E}_{\ell,g}^{(+)}(g) = 4\left(n + g + \frac{1}{2}\right)\omega, \quad\quad \phi_{\ell,n}^{(+)}(x; g) = e^{-\frac{1}{2}\omega x^2}x^{g+\ell+1}L_n^{(g+\ell+\frac{1}{2})}(\eta).$ (17)

The eigenfunctions $\phi_{\ell,n}^{(-)}(x; g)$ of $\mathcal{H}_\ell^{(-)}(g)$ are obtained by applying the operator $\mathcal{A}_\ell^+(g)$ on $\phi_{\ell,n}^{(+)}(x; g)$: $\phi_{\ell,n}^{(-)}(x; g) = \mathcal{A}_\ell^+(g)\phi_{\ell,n}^{(+)}(x; g)$. It turns out that $\phi_{\ell,n}^{(-)}(x; g)$, up to multiplicative constants, are just the eigenfunctions $\phi_{\ell,n}(\eta; g)$ in (9) and (10) of the Sasaki-Odake Hamiltonian.

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III. DIRAC EQUATION WITH MAGNETIC FIELD IN CYLINDRICAL COORDINATES

Having reviewed the deformed radial oscillators associated with the exceptional X_ℓ Laguerre polynomials, we will like to see if there are physical systems in which these new polynomials could play a role. It turns out that there are indeed such systems. In what follows we shall indicate some of them.

First let us consider the Dirac equation in 2+1 dimensions coupling minimally with a cylindrically symmetric magnetic field. The discussion can be extended to 3+1 dimensions where the magnetic field does not depend on the variable z. The Dirac equation for a unit positive ($q = 1$) charged particle minimally coupled to an magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ has the form (we set $c = \hbar = 1$)

$$H_D \Psi(\mathbf{r}) = E \Psi(\mathbf{r})$$
$$H_D = \boldsymbol{\sigma} \cdot (\mathbf{p} - \mathbf{A}) + \sigma_3 M, \tag{18}$$

where H_D is the Dirac Hamiltonian, $\boldsymbol{p} = -i\nabla$ is the momentum operator, E and M are energy and rest mass of the particle, and $\boldsymbol{\sigma}$ are the Pauli matrices.

We shall consider magnetic field which is cylindrically symmetric. Then the vector potential has only ϕ-component:

$$\boldsymbol{A}(\mathbf{r}) = A_\phi(r)\hat{\phi}, \quad r = |\boldsymbol{r}|. \tag{19}$$

Instead of the variable x, we shall use the conventional notation r for the radial variable here, and in Sect. IV.A and B. The magnetic field is

$$B_3(r) = \frac{1}{r}\frac{d}{dr}\left(r A_\phi(r)\right). \tag{20}$$

The wave function is taken to have the form

$$\psi_m(r, \phi) = \begin{pmatrix} f_+(r)e^{im\phi} \\ -if_-(r)e^{i(m+1)\phi} \end{pmatrix} \tag{21}$$

with integral number m. The function $\psi_m(r, \phi)$ is an eigenfunction of the conserved total angular momentum $J_3 = L_3 + S_3 = -i\partial/\partial\phi + \sigma_3/2$ with eigenvalue $j = m + 1/2$. It should be reminded that m is not a good quantum number. This is evident from the fact that the

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two components of ψ_m depend on the integer m in an asymmetric way. Only the eigenvalues j of the conserved total angular momentum J_3 are physically meaningful.

From the identities

$$\boldsymbol{\sigma} \cdot \mathbf{p} = i(\boldsymbol{\sigma} \cdot \hat{\mathbf{r}}) \left(-\partial_r + \frac{1}{r}(\boldsymbol{\sigma} \cdot \mathbf{L}) \right), \tag{22}$$

$$\boldsymbol{\sigma} \cdot \mathbf{A} = i(\boldsymbol{\sigma} \cdot \hat{\mathbf{r}})\sigma_3 A_\phi, \tag{23}$$

one gets

$$\boldsymbol{\sigma} \cdot (\mathbf{p} - \mathbf{A})) = i(\boldsymbol{\sigma} \cdot \hat{\mathbf{r}}) \left(-\partial_r + \frac{1}{r}\left(\sigma_3 J_3 - \frac{1}{2} \right) - \sigma_3 A_\phi \right). \tag{24}$$

Upon using the relation

$$(\boldsymbol{\sigma} \cdot \hat{\mathbf{r}}) \begin{pmatrix} F e^{im\phi} \\ G e^{i(m+1)\phi} \end{pmatrix} = \begin{pmatrix} G e^{im\phi} \\ F e^{i(m+1)\phi} \end{pmatrix}, \tag{25}$$

one can reduce (18) to

$$\left(\frac{d}{dr} - \frac{m + \frac{1}{2}}{r} + A_\phi \right) f_+ = (E + M) f_-, \tag{26}$$

$$\left(-\frac{d}{dr} - \frac{m + \frac{1}{2}}{r} + A_\phi \right) f_- = (E - M) f_+. \tag{27}$$

This shows that f_+ and f_- forms a one-dimensional SUSY pairs. If we take the prepotential W such that

$$W' = \frac{g}{r} - A_\phi, \quad g \equiv m + \frac{1}{2}, \tag{28}$$

and $A^\pm = \pm d/dr - W'$ (as usual, the prime here denotes derivatives with respect to the basic variable, which is r in this case), then eqs.(26) and (27) become

$$A^- A^+ f_+ = \left(E^2 - M^2 \right) f_+, \tag{29}$$

$$A^+ A^- f_- = \left(E^2 - M^2 \right) f_-, \tag{30}$$

Explicitly, the above equations read

$$\left(-\frac{d^2}{dr^2} + W'^2 \pm W'' \right) f_\pm = \left(E^2 - M^2 \right) f_\pm. \tag{31}$$

The ground state, with $E^2 = M^2$, is given by one of the following two sets of equations:

$$A^+ f_+^{(0)}(r) = 0, \quad f_-^{(0)}(r) = 0, \tag{32}$$

$$A^- f_-^{(0)}(r) = 0, \quad f_+^{(0)}(r) = 0, \tag{33}$$

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depending on which solution is normalizable. The solutions are generally given by

$$f_{\pm}^{(0)} \propto r^{\pm(m+\frac{1}{2})} \exp\left(\mp \int dr A_\phi\right). \tag{34}$$

To be specific, we consider the situation where $m \geq 0$ and $\int dr A_\phi > 0$, so that $f_+^{(0)}$ is normalizable, and $f_-^{(0)} = 0$. The other situation can be discussed similarly.

Eq. (28) relates $A_\phi(r)$ and $W(r)$. This gives a way to obtain A_ϕ that defines exactly solvable model. Particularly, from the Table (4.1) in [4], one concludes that there are three forms of A_ϕ giving exact solutions of the problem based on the conventional classical orthogonal polynomials:

   i) oscillator-like : $A_\phi(r) \propto r$;

   ii) Coulomb potential-like : $A_\phi(r) \propto$ constant ;

   iii) zero field-like : $A_\phi(r) \propto 1/r$.

Case (i) corresponds simply to the well-known Landau level problem.

Now with the discovery of the exceptional Laguerre polynomials, one can find an infinite family of vector potentials A_ϕ that give the oscillator-like spectra. These are the deformed Landau systems. There are, however, two different types of deformed Landau systems, depending on whether we choose A^-A^+ in (29) to correspond to $H_\ell^{(+)}(g)$ in (6), or to $\mathcal{H}_\ell^{(+)}(g)$ in (12), or equivalently, (14).

If we choose A^-A^+ to correspond to $H_\ell^{(+)}(g)$, then from (28) and (7) we get the required vector potential (we add a superscript to indicate its family ℓ)

$$\begin{aligned}
A_\phi^{(\ell)}(r) &= \frac{m+\frac{1}{2}}{r} - W_\ell', \\
&= \omega r - \left[\frac{\ell}{r} + \frac{\xi_\ell'(\eta; g+1)}{\xi_\ell(\eta; g+1)} - \frac{\xi_\ell'(\eta; g)}{\xi_\ell(\eta; g)}\right]. \tag{35}
\end{aligned}$$

As before, the ξ_ℓ is given by (8) for the L1 and the L2 case. These deformed systems are iso-spectral to the relativistic Landau system in case (i) mentioned before: the eigen-energies being

$$\mathcal{E}_{\ell,n}(g) = E_{\ell,n}^2 - M^2 = 4n\omega, \tag{36}$$

which is independent of g and ℓ. The eigenfunction f_+ is given by $\phi_{\ell,n}(r; g)$ in (9): $f_+ \propto \phi_{\ell,n}(r; g)$, and f_- is obtained from f_+ by (26): $f_- \propto A^+ f_+$. Thus both the upper and lower components of the eigenfunctions are related to the exceptional orthogonal polynomials in this case.

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In the other choice of the vector potential, i.e., with $A^- A^+$ corresponding to $\mathcal{H}_\ell^{(+)}(g)$, only the lower component f_- involves the new polynomials. This time $A_\phi^{(\ell)}(r)$ is determined by (28) and (11):

$$A_\phi^{(\ell)}(r) = \begin{cases} -\omega r - \frac{\ell-1}{r} - \frac{\xi_\ell'(\eta;g)}{\xi_\ell(\eta;g)} & : \text{L1} \\ \omega r + \frac{2g+\ell}{r} - \frac{\xi_\ell'(\eta;g)}{\xi_\ell(\eta;g)} & : \text{L2}. \end{cases} \tag{37}$$

The eigen-energies are given by $\mathcal{E}_{\ell,n}^{(+)}(g)$ in (16) and (17),

$$\mathcal{E}_{\ell,n}^{(+)}(g) = E_{\ell,n}^2 - M^2 = \begin{cases} 4\left(n + g + 2\ell - \frac{1}{2}\right)\omega & : \text{L1} \\ 4\left(n + g + \frac{1}{2}\right)\omega & : \text{L2}. \end{cases} \tag{38}$$

The eigenfunction f_+ is given by $\phi_{\ell,n}^{(+)}(r;g)$ in (16) and (17): $f_+ \propto \phi_{\ell,n}^{(+)}(x;g)$, and f_- is again obtained from f_+ by (26): $f_- \propto \phi_{\ell,n}^{(-)} = \mathcal{A}_\ell^+ f_+$. Hence, f_- is related to the new exceptional orthogonal polynomial, as discussed in the statements below (17). Note here that in this case the ground state energy is not zero, and hence f_- need not vanish. Such situation is called broken supersymmetry in SUSY quantum mechanics [4].

The above discussion can be straightforwardly extended to the case of two-dimensional Pauli equation. This is so as the square of the Dirac equation is related to the Pauli equation. Specifically, we have $H_D^2 = H_P + M^2$, where the Pauli Hamiltonian H_P is

$$H_P \equiv (\mathbf{p} - \mathbf{A})^2 - \sigma_3(\nabla \times \boldsymbol{A})_3. \tag{39}$$

IV. DIRAC EQUATIONS WITH NON-MINIMAL COUPLING

The example discussed in the previous section illustrates how physical Dirac systems whose eigenfunctions are related to the exceptional orthogonal polynomials can be constructed. The construction relies mainly on the fact that the two radial components of the wave function form a SUSY pair as in (26) and (27). Thus, as long as a Dirac equation can be reduced to such a form, this construction applies and one can obtain new Dirac systems that involve the new polynomials. In this section we indicate three more Dirac-type equations which allow such construction. Unlike the the model discussed in Sect. III, these three systems involve non-minimal couplings between the fermion and the external electromagnetic fields.

A. Dirac-Pauli equation with electric field in spherical coordinates

Consider a neutral fermion interacting with electromagnetic fields through its anomalous magnetic moment. The relativistic wave equation that describe such interaction is called the Dirac-Pauli equation [14, 15]. As shown in [15], this equation includes as a special case the so-called Dirac oscillator, which has attracted much attention in recent years [16].

In this section we shall consider the case where the external field is purely electrical. The Dirac-Pauli equation that describes the motion of a neutral fermion of spin-1/2 with mass M and an anomalous magnetic moment μ in an external electric field $\mathbf{E}$ is given by $H_{DP}\Psi = E\Psi$, with the Hamiltonian

$$H_{DP} = \boldsymbol{\alpha} \cdot \mathbf{p} + i\mu\beta\boldsymbol{\alpha} \cdot \mathbf{E} + \beta M. \tag{40}$$

Here $\boldsymbol{\sigma}$ and β are the Dirac matrices which in the standard representation are given by

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{41}$$

where $\boldsymbol{\sigma}$ are the Pauli matrices. We also define $\Psi = (\chi, \varphi)^t$, where t denotes transpose, and both χ and φ are two-component spinors. Then the Dirac–Pauli equation becomes

$$\begin{aligned} \boldsymbol{\sigma} \cdot (\mathbf{p} - i\mu\mathbf{E})\chi &= (E + M)\varphi, \\ \boldsymbol{\sigma} \cdot (\mathbf{p} + i\mu\mathbf{E})\varphi &= (E - M)\chi. \end{aligned} \tag{42}$$

This exhibits the intrinsic SUSY structure of the system. As in the previous case, this allows us to construct exactly solvable Dirac-Pauli equations which are related to the exceptional polynomials. This we shall do below in the spherical and cylindrical coordinates.

First let us consider central electric field $\mathbf{E} = E_r(r)\hat{\mathbf{r}}$. In this case, one can choose a complete set of observables to be $\{H, \mathbf{J}^2, J_z, \mathbf{S}^2 = 3/4, K\}$. Here $\mathbf{J}$ is the total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$, where $\mathbf{L}$ is the orbital angular momentum, and $\mathbf{S} = \frac{1}{2}\boldsymbol{\Sigma}$ is the spin operator. The operator K is defined as $K = \gamma^0(\boldsymbol{\Sigma} \cdot \mathbf{L} + 1)$, which commutes with both H and $\mathbf{J}$. Explicitly, we have

$$\begin{aligned} K &= \text{diag}\left(\hat{k}, -\hat{k}\right), \\ \hat{k} &= \boldsymbol{\sigma} \cdot \mathbf{L} + 1. \end{aligned} \tag{43}$$

The common eigenstates can be written as [15]

$$\psi = \frac{1}{r} \begin{pmatrix} f_+(r)\mathcal{Y}^k_{jm_j} \\ if_-(r)\mathcal{Y}^{-k}_{jm_j} \end{pmatrix}, \tag{44}$$

here $\mathcal{Y}^k_{jm_j}(\theta, \phi)$ are the spin harmonics satisfying

$$\mathbf{J}^2\mathcal{Y}^k_{jm_j} = j(j+1)\mathcal{Y}^k_{jm_j}, \quad j = \frac{1}{2}, \frac{3}{2}, \ldots, \tag{45}$$

$$J_z\mathcal{Y}^k_{jm_j} = m_j\mathcal{Y}^k_{jm_j}, \qquad |m_j| \le j, \tag{46}$$

$$\hat{k}\mathcal{Y}^k_{jm_j} = -k\mathcal{Y}^k_{jm_j}, \qquad k = \pm(j + \frac{1}{2}), \tag{47}$$

and

$$(\boldsymbol{\sigma} \cdot \hat{\mathbf{r}})\mathcal{Y}^k_{jm_j} = -\mathcal{Y}^{-k}_{jm_j}. \tag{48}$$

Using the identity (22), one gets

$$\boldsymbol{\sigma} \cdot (\mathbf{p} \pm \mathbf{i}\mu\mathbf{E}) = i(\boldsymbol{\sigma} \cdot \hat{\mathbf{r}})\left(-\partial_r + \frac{\hat{k}-1}{r} \pm \mu E_r\right). \tag{49}$$

Eq.(42) then reduces to

$$\left(\frac{d}{dr} + \frac{k}{r} + \mu E_r\right) f_+ = (E + M) f_-, \tag{50}$$

$$\left(-\frac{d}{dr} + \frac{k}{r} + \mu E_r\right) f_- = (E - M) f_+. \tag{51}$$

These two equations have the same SUSY structure as (26) and (27), and so one can proceed as in the last section. Again, we consider the situation where $k < 0$ and $\int dr\mu E_r > 0$, so that for unbroken supersymmetry, the ground state has upper component $f_+^{(0)}$ normalizable, and lower component $f_-^{(0)} = 0$. The other situation can be discussed similarly. In this case, eq.(28) becomes

$$W' = \frac{g}{r} - \mu E, \quad g \equiv |k|. \tag{52}$$

Thus all the discussions following (28) can be carried over with the change $A_\phi(r) \to \mu E_r(r)$.

Particularly, the case with $\mu E_r \propto r$ is just the Dirac oscillator [16]. Thus the new exactly solvable models associated with the X_ℓ Laguerre polynomials constructed according to the procedure in Sect. III are the new deformed Dirac oscillators.

B. Dirac-Pauli equation with electric fields in cylindrical coordinates

We now turn to $2+1$ dimensions where the electric field is cylindrically symmetric. This discussion can be readily extended to the case in $3+1$ dimensions where the electric field is constant along the z direction.

The Dirac-Pauli Hamiltonian takes the form

$$H_{DP} = \boldsymbol{\sigma} \cdot \mathbf{p} + i\mu\sigma_3\boldsymbol{\sigma} \cdot \mathbf{E} + \sigma_3 M. \tag{53}$$

All vectors lie in the $x-y$ plane. For $\mathbf{E} = E_r(r)\hat{\mathbf{r}}$, the second term in (53) is $-i\mu(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})\sigma_3 E_r(r)$ (note that $\sigma_3 \cdot \boldsymbol{\sigma} = -\boldsymbol{\sigma} \cdot \sigma_3$ in this case). Comparing this term with $-\boldsymbol{\sigma} \cdot \mathbf{A}$ in (18) and (23), one sees that the Hamiltonians (53) and (18) are equivalent if we make the correspondence $\mu E_r(r) \leftrightarrow A_\phi(r)$. Therefore, the results in Sect. III can be directly carried over to the present case.

We note here that the correspondence $\mu E_r(r) \leftrightarrow qA_\phi(r)$ (we restore the charge q) also underlies the duality between the Aharonov–Casher effect [17], which is described by the Dirac-Pauli equation, and the well-known Aharonov–Bohm effect [18], described by the Dirac equation with minimal coupling. The Aharonov–Casher effect concerns the topological phase experienced by a neutral fermion with a magnetic moment when diffracted around a line of electric charge. It is an electrodynamic and quantum–mechanical dual of the Aharonov–Bohm effect, which gives a phase shift for a charged particle diffracting around a tube of magnetic flux.

C. Two-dimensional Dirac equation with Lorentz scalar potential

Let us consider a $(1 + 1)$-dimensional Dirac Hamiltonian of the kind

$$H = \alpha p + \beta(M + V_s(x)) \tag{54}$$

where M is the mass of the fermion, $p = -id/dx$, α and β are the Dirac matrices, and V_s is the Lorentz scalar potential. Such model is of interest in the theory of nuclear shell model [19], and as a model of the self-compatible field of a quark system [20].

This system is supersymmetric [21], as can be easily shown as follows. We represent the Dirac matrices by

$$\alpha = \sigma_2, \quad \beta = \sigma_1. \tag{55}$$

Then the Dirac equation $H\psi = E\psi$ for the two-component wave function

$$\psi(x) = \begin{pmatrix} \psi_+(x) \\ \psi_-(x) \end{pmatrix} \tag{56}$$

takes the form

$$\left(\frac{d}{dx} - W'(x)\right)\psi_+ = E\psi_-,$$
$$\left(-\frac{d}{dx} - W'(x)\right)\psi_- = E\psi_+. \tag{57}$$

Here $W'(x) \equiv -(V_s(x) + M)$ and $\mathcal{E} = E^2$. Eq. (57) is now in the SUSY form, with $W(x)$ playing the role of the prepotential. As such, following the procedure in Sect. III, we can obtain new exactly solvable systems related to the exceptional Laguerre polynomials.

In fact, in this case the domain need not be confined to the half-line. Thus we can construct new solvable Dirac systems whose eigenfunctions are related to the exceptional Jacobi polynomials by simply linking W with the prepotential corresponding to exceptional Jacobi polynomials.

V. FOKKER-PLANCK EQUATIONS

Finally, we discuss briefly how the exceptional orthogonal polynomials can appear in the Fokker-Planck (FP) equations. In one dimension, the FP equation of the probability density $\mathcal{P}(x,t)$ is [22]

$$\frac{\partial}{\partial t}\mathcal{P}(x,t) = \mathcal{L}\mathcal{P}(x,t),$$
$$\mathcal{L} \equiv -\frac{\partial}{\partial x}D^{(1)}(x) + \frac{\partial^2}{\partial x^2}D^{(2)}(x). \tag{58}$$

The functions $D^{(1)}(x)$ and $D^{(2)}(x)$ in the FP operator $\mathcal{L}$ are, respectively, the drift and the diffusion coefficient (we consider only time-independent case). The drift coefficient represents the external force acting on the particle, while the diffusion coefficient accounts for the effect of fluctuation. Without loss of generality, in what follows we shall take $D^{(2)} = 1$. The drift coefficient can be defined by a prepotential $W(x)$ as $D^{(1)}(x) = 2W'(x)$.

The FP equation is closely related to the Schrödinger equation [22, 23]. Substituting

$$\mathcal{P}(x,t) \equiv e^{-\lambda t}e^{W(x)}\phi(x). \tag{59}$$

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into the FP equation, we find that ϕ satisfies the Schrödinger-like equation: $H\phi = \lambda\phi$, where

$$
\begin{aligned}
H &\equiv -e^{-W}\mathcal{L}e^{W} \\
&= -\frac{\partial^2}{\partial x^2} + W'(x)^2 + W''(x).
\end{aligned}
$$

Thus ϕ satisfies the time-independent Schrödinger equation with Hamiltonian H and eigenvalue λ, and $\phi_0 = \exp(W)$ is the zero mode of H: $H\phi_0 = 0$.

It is now clear that FP equations transformable to exactly solvable Schrödinger equations can be exactly solved. If all the eigenfunctions ϕ_n ($n = 0, 1, 2, \ldots$) of H with eigenvalues λ_n are solved, then the eigenfunctions $\mathcal{P}_n(x)$ of $\mathcal{L}$ corresponding to the eigenvalue $-\lambda_n$ is $\mathcal{P}_n(x) = \phi_0(x)\phi_n(x)$. The stationary distribution is $\mathcal{P}_0 = \phi_0^2 = \exp(2W)$ (with $\int \mathcal{P}_0(x)\,dx = 1$), which is obviously non-negative, and is the zero mode of $\mathcal{L}$: $\mathcal{L}\mathcal{P}_0 = 0$. Any positive definite initial probability density $\mathcal{P}(x, 0)$ can be expanded as $\mathcal{P}(x, 0) = \phi_0(x)\sum_n c_n\phi_n(x)$, with constant coefficients c_n ($n = 0, 1, \ldots$)

$$
c_n = \int_{-\infty}^{\infty} \phi_n(x)\left(\phi_0^{-1}(x)\mathcal{P}(x, 0)\right)dx. \tag{60}
$$

Then at any later time t, the solution of the FP equation is $\mathcal{P}(x, t) = \phi_0(x)\sum_n c_n\phi_n(x)\exp(-\lambda_n t)$.

Thus all the shape-invariant potentials in supersymmetric quantum mechanics give the corresponding exactly solvable FP systems. One needs only to link the prepotential W in the Schrödinger system with the drift potential corresponding to the drift coefficient $D^{(1)} = 2W'$ in the FP system. For example, the shifted oscillator potential in quantum mechanics corresponds to the FP equation for the well-known Ornstein–Uhlenbeck process [22].

Of interest to us here is the FP equation that corresponds to the radial oscillator potential. This FP equation describes the so-called Rayleigh process [24]. When $W(x)$ is replaced by the the prepotential (7), we obtain an exactly solvable FP equation, describing a deformed Rayleigh process, whose eigenfunctions are given by the exceptional X_ℓ Laguerre polynomials.

Again, as with the Dirac equation with Lorentz scalar potential, in this case one can get exactly solvable FP equations whose eigenfunctions are related to the exceptional Jacobi polynomials by simply linking W with the prepotential corresponding to exceptional Jacobi polynomials.

VI. SUMMARY

The discovery of the exceptional X_ℓ Laguerre and Jacobi polynomials has opened up new avenues in the area of mathematical physics and in classical analysis. Unlike the well-known classical orthogonal polynomials which start with constant terms, these new polynomials have lowest degree $\ell = 1, 2, \ldots$, and yet they form complete set with respect to some positive-definite measure. Some properties of these new polynomials have been studied, and many more have yet to be investigated.

In this paper we have presented some physical models in which these new polynomials could play a role. For clarity of presentation, we concentrate mainly on the exceptional Laguerre polynomials. We show how some Dirac equations coupled minimally and non-minimally with external fields, and the Fokker-Planck equations can be exactly solvable with the exceptional Laguerre polynomials as the main part of the eigenfunctions. The systems presented here have enlarged the number of exactly solvable physical systems known so far.

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